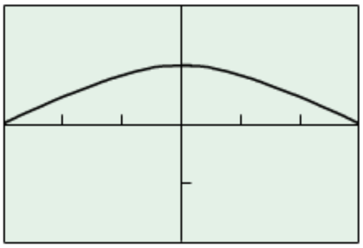
EXPLORATION 1 Making a Conjecture by Graphing the Derivative

Graph $y_1 = \sin(x)$ and $y_2 = \frac{d}{dx}(\sin x)$ in the window $[-2\pi, 2\pi]$ by $[-4, 4]$ (Figure 3.35). Use the numerical derivative to graph $y_2 = \frac{d}{dx}(\sin x)$. See page 113.

1. When the graph of $y_1 = \sin x$ is increasing, what is true about the graph of $y_2 = \frac{d}{dx}(\sin x)$? positive
2. When the graph of $y_1 = \sin x$ is decreasing, what is true about the graph of $y_2 = \frac{d}{dx}(\sin x)$? negative
3. When the graph of $y_1 = \sin x$ stops increasing and starts decreasing, what is true about the graph of $y_2 = \frac{d}{dx}(\sin x)$? = zero
   goes from positive to negative
4. At the places where $y_2 = \frac{d}{dx}(\sin x) = \pm 1$, what appears to be the slope of the graph of $y_1 = \sin x$? ± 1
5. Make a conjecture about what function the derivative of sine might be. Test your conjecture by graphing your function and $y_2 = \frac{d}{dx}(\sin x)$ in the same viewing window.
6. Now let $y_1 = \cos x$ and $y_2 = \frac{d}{dx}(\cos x)$. Answer questions (1) through (5) *without* looking at the graph of $y_2 = \frac{d}{dx}(\cos x)$ until you are ready to test your conjecture about what function the derivative of cosine might be.



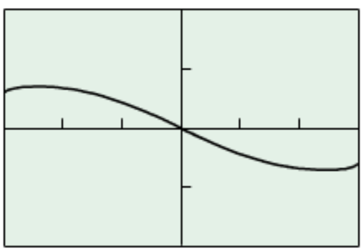
[–3, 3] by [–2, 2]
(a)

X	Y1
–.03	.99985
–.02	.99993
–.01	.99998
0	ERROR
.01	.99998
.02	.99993
.03	.99985

Y1 = sin(X)/X

(b)

Figure 3.36 (a) Graphical and (b) tabular support that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.



[–3, 3] by [–2, 2]
(a)

X	Y1
–.03	.015
–.02	.01
–.01	.005
0	ERROR
.01	–.005
.02	–.01
.03	–.015

Y1 = (cos(X)–1)/X

(b)

Figure 3.37 (a) Graphical and (b) tabular support that $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$.

Confirm Analytically

Prove analytically the derivative of sin x and the derivative of cos x.

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

EXAMPLE 1 Revisiting the Differentiation Rules

Find the derivatives of **(a)** $y = x^2 \sin x$ and **(b)** $u = \cos x/(1 - \sin x)$.

a) $y' = x^2(\cos x) + \sin x\,(2x)$

$= x^2\cos x + 2x\sin x$

b) $u' = \dfrac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$

$= \dfrac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$ $= 1$

$= \dfrac{\cancel{1-\sin x}}{(1-\sin x)^{\cancel{2}}}$

$= \dfrac{1}{1-\sin x}$

EXAMPLE 2 The Motion of a Weight on a Spring

A weight hanging from a spring (Figure 3.38) is stretched 5 units beyond its rest position $(s = 0)$ and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t? Describe its motion.

$v(t) = s'(t) = -5 \sin t$

$a(t) = v'(t) = -5 \cos t$

DEFINITION Jerk

Jerk is the derivative of acceleration. If a body's position at time t is $s(t)$, the body's jerk at time t is

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

EXAMPLE 3 A Couple of Jerks

(a) The jerk caused by the constant acceleration of gravity ($g = -32$ ft/sec^2) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

This explains why we don't experience motion sickness while just sitting around.

(b) The jerk of the simple harmonic motion in Example 2 is

$j(t) = a'(t) = 5\sin t$

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x},$$
$$\cot x = \frac{\cos x}{\sin x}, \quad \csc x = \frac{1}{\sin x}$$

With your group, analytically determine the derivatives of the above functions.

$\tan x$: $\dfrac{\cos x(\cos x) - \sin x(\cos x)}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

$\cot x$: $\dfrac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$

$= \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \dfrac{-1}{\sin^2 x} = -\csc^2 x$

$\sec x$: $\dfrac{\cos x(0) - 1(-\sin x)}{\cos^2 x}$

$= \dfrac{\sin x}{(\cos x)(\cos x)} = \tan x \cdot \dfrac{1}{\cos x}$

$= \tan x \sec x$

$= \sec x \tan x$

$\csc x$: $\dfrac{\sin x(0) - 1(\cos x)}{\sin^2 x}$

$= \dfrac{-\cos x}{\sin x \sin x} = -\cot x \cdot \dfrac{1}{\sin x}$

$= -\cot x \csc x$

$= -\csc x \cot x$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

EXAMPLE 4 Finding Tangent and Normal Lines

Find equations for the lines that are tangent and normal to the graph of

$$f(x) = \frac{\tan x}{x}$$

at $x = 2$. Support graphically.

$$y_1 = \frac{\tan x}{x}$$

Use the numerical derivative on your calculator to find f'(2).

slope, point

≈ 3.433 $(2, -1.093)$

$\frac{d}{dx} y_1 \quad x=2$ $y_1(2)$

tangent: $y + 1.093 = 3.433(x-2)$

normal: $y + 1.093 = -0.291(x-2)$

$\left(-\frac{1}{3.433}\right)$

EXAMPLE 5 A Trigonometric Second Derivative

Find y'' if $y = \sec x$.

$$y' = \underline{\sec x}\ \underline{\tan x}$$

$$y'' = \sec x\,(\sec^2 x) + \tan x\,(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$